RAMANUJAN AND PARTITIONS

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On the occasion of the 112-th birth anniversary of Srinivasa Ramanujan, Krishnaswami Alladi pays tribute by describing the pathbreaking contributions of the Indian genius to the theory of partitions. Since we are also at the eve of a new millenium, the author discusses how Ramanujan's work in this area will continue to have influence in the future. - Editor

Although the theory of partitions was founded by Euler two centuries ago, it is no exageration to say that it was Ramanujan's spectacular contributions to this field at the beginning of this century that propelled it to a glorious position that it deserves and continues to occupy today. Ramanujan's work on partitions provided many unexpected and important connections with other areas such as number theory, combinatorics, analysis, computer algebra and physics, and so this subject is now being studied from several points of view. In the past few decades, Professor George Andrews of The Pennsylvania State University has systematically studied a wide class of problems in the theory of partitions and written an Encylopedia on Partitions - the standard reference in this field. Professor Andrews is thus the torch bearer of the subject today, and to him also goes the credit of our present understanding of many of Ramanujan's identities in the context of partitions. Indeed, to many current researchers like me in this field, Ramanujan is the inspiration, and Andrews the medium to understand Ramanujan's work and its ramifications. In this article I shall discuss some of Ramanujan's most significant contributions to partitions and the progress in several directions it led to. I will also indicate how Ramanujan's work will form the basis for future research. In preparing this article, I have primarily depended on Hardy's classic twelve lectures on Ramanujan, Andrews' Encyclopedia and his CBMS Lectures on partitions, q-series, and allied fields, as well as several stimulating conversations and collaborations with Professors George Andrews and Basil Gordon (University of California, Los Angeles), in addition to my own research in this area.

Partitions: By a partition of a positive integer, we mean a representation of that integer as a sum of positive integers. Two partitions are considered the same if they differ only in the order of their parts. For example, 3+2+1+1 and 3+1+2+1 are the same partition of 7. Denote by $p(n)$, the number of (unrestricted) partitions of n. For example, $p(5) = 7$ because there are 7 partitions of 5, namely, 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, and 1+1+1+1+1. There is a technique called the *method of generating functions* whereby one can translate all the information in a sequence of numbers into a single continuous function from the realm of calculus. Euler noticed that that the generating function of $p(n)$ possessed a beautiful product representation. Euler also realised, that using generating functions, elegant results on partitions could be proved such as: *The number of partitions of an integer into odd parts equals the number of partitions of that integer into distinct (non-repeating) parts.* This is commonly referred to as Euler's theorem. For example, there are six partitions of 8 into odd parts, namely, 7+1, 5+3, 5+1+1+1, 3+3+1+1, 3+1+1+1+1+1, and 1+1+...+1. There are also six partitions of 8 into distinct parts, namely, 8, 7+1, 6+2, 5+3, 5+2+1, and 4+3+1. Euler also obtained a very useful recurrence relation for partitions by means of his famous *pentagonal numbers theorem.* With the foundation laid by

Euler, the theory of partitions underwent a glorious transformation with the magic touch of Ramanujan. The Indian genius produced a variety of new results which showed connections with many different areas. These results were also startlingly beautiful. I will now explain a few of Ramanujan's discoveries, and for each, I shall describe the very interesting history, give some idea of current research, and indicate future directions.

The Hardy-Ramanujan formula: In one of his letters to Hardy written in 1913, Ramanujan gave a formula for the coefficients of a certain series expansion of an infinite product which suggested that there ought to be a similar exact formula for the partition function $p(n)$ in terms of continuous functions. Hardy felt that this claim of Ramanujan was too good to be true, but was convinced that it was possible to develop an asymptotic formula for $p(n)$ in terms of continuous functions. Asymptotic formulas may be thought of as approximate formulas. Generally they do not give an exactly correct answer, but they are close. By an ingenious and intricate calculation involving the singularities of the generating function of $p(n)$ in the unit circle, Hardy and Ramanujan obtained an asymptotic formula, which when calculated up to a certain number of terms, yielded a value which differed from $p(n)$ by a quantity no more than the reciprocal of the square root of n. Since $p(n)$ is an integer, it is clear that the exact value of $p(n)$ is the nearest integer value to what is given by the series. This was indeed amazing, and Hardy wanted to test the correctness of the result. So he asked his friend Major MacMahon to compute the values of $p(n)$ using Euler's recurrence formula. It turned out that the value $p(200) = 3972999029388$ given by Euler's recurrence, was also given by the Hardy-Ramanujan formula! The series representation of Hardy-Ramanujan is genuinely an asymptotic series in the sense when summed up to infinity, it diverges. Subsequently Hans Rademacher noticed that by making a very mild but important change, namely, by replacing the exponential functions with hyperbolic functions, the Hardy-Ramanujan asymptotic series could be converted into a series that in fact converges to $p(n)$. Actually, in the 1913 letter to Hardy, Ramanujan used hyperbolic functions to claim an exact formula for a related problem, and so Ramanujan was indeed correct in surmising that a similar exact formula would exist for $p(n)$. Professor Atle Selberg of the Institute for Advanced Study in Princeton, one of the greatest living mathematicians, said during his address at the Ramanujan Centennial in Madras on December 22, 1987, that the exact formula that Rademacher obtained was actually more natural than the Hardy-Ramanujan formula. Indeed Selberg discovered this exact formula on his own, but did not publish it once he found out that Rademacher had done it earlier. An aspect of Ramanujan's discoveries that comes up time and again is the surprising and unbelievable form of the results. The exact formula for $p(n)$ that Ramanujan conjectured was considered unbelievably good by Hardy who settled for less - namely an asymptotic formula, and Ramanujan agreed (according to Selberg) out of respect for his mentor. In any case, the Hardy-Ramanujan asymptotic formula gave rise to very powerful analytic method to evaluate the coefficients of series arising in a wide class of problems in additive number theory. This *circle method* originally due to Hardy-Ramanujan and subsequently developed by Hardy-Littlewood and others, is one of the most widely applicable methods today, and will continue to be a major tool in the future.

Ramanujan congruences: As soon as Ramanujan saw the table of values of the partition function that MacMahon had prepared (in order to check the Hardy-Ramanujan asymptotic formula) he (Ramanujan) wrote down three congruences. Hardy was stunned in disbelief when he saw these claims. The first Ramanujan congruence states that *the number of partitions of an integer of the form 5n+4 is always a multiple of 5.* The second congruence states that *the number of partitions of 7n+5 is a multiple of 7.* The third congruence is the statement that *the number of partitions of 11n+6 is a multiple of 11.* What stunned Hardy was there are divisibility properties for partitions which are combinatorial odjects defined by an additive process.

In the table of partitions that MacMahon had prepared, the values of $p(n)$ are listed in columns of length 5 starting with the value $p(0) = 1$ as shown below.

n	$p(n)$
0	1
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30

....

Thus the values $p(5n+4)$ are at the bottom of each column, and the lower right hand digit in each block is either 0 or 5. So in principle, any one staring at the last entry of each column could have observed Ramanujan's first congruence, but one has to be in search of such a property in order to observe it. MacMahon prepared the table and Hardy checked it but neither of them observed the congruence because they were not looking for such surprising connections! Ramanujan who always had the eye for the unexpected, wrote the congruence down as soon as he saw the table.

The study of such divisibility properties or congruences for partition functions and similar objects has been one of the most fruitful and active areas of research in number theory during this century. What is fascinating is the role of the theory of modular forms (an abstract and beautiful area involving analysis, algebra and number theory) in the study of such congruences.

Although modular forms have been the main tool used in the study of such congruences, combinatorial explanations are of great interest because partitions are combinatorial objects. In 1944, Freeman Dyson, then a young student in Cambridge University, found such an explanation of Ramanujan's first two congruences using the concept of *rank of a partition.* Dyson published this in the undergraduate mathematics journal of Cambridge called *Eureka,* and conjectured that a similar explanation ought to exist for the third congruence by means of a different statistic that he called the *crank,* that had not been found yet. Professor Dyson has humourously remarked that this was perhaps the first instance in mathematics where an object had been named before it was found. Dyson's crank remained elusive for many years, but in 1987, immediately after the Ramanujan Centennial Conference at the

University of Illinois, Frank Garvan and George Andrews found the crank at last. Frank Garvan is now my colleague at the University of Florida.

A recent pathbreaking work in this field is due to Professor Ken Ono of The Pennsylvania State University and the University of Wisconsin. It had been widely believed that the values of the partition function $p(n)$ behave randomly modulo m except for the Ramanujan congruences. Ramanujan had conjectured that if $m \neq 5, 7, 11$, then in every arithmetic progression there are infinitely many values of the partition function which are not a multiples of m. Last year, Ono showed that Ramanujan's conjecture is mostly correct. More precisely, Ono proved that for every prime $m \neq 5, 7, 11$ the proportion of arithmetic progressions for which Ramanujan's conjecture holds is greater than $1 - 10^{-100}$. Although this seems like phenomenal evidence, very recently, Ono has shown that Ramanujan's conjecture is actually false! That is by a deep study involving L-functions and Galois representations, Ono showed that for every $m \geq 13$ there are arithmetic progressions with very large moduli, in which the values $p(n)$ are always multiples of m. Thus the Ramanujan type congruences although very rare are indeed plentiful. Thus the study of Ramanujan type congruences not only for the partition function but for coefficients of modular forms will continue to be a very active field of research.

Rogers-Ramanujan identities: In the entire theory of partitions and q-series, the Rogers-Ramanujan identities are unmatched in simplicity elegance and depth. The statement of the first identity is: *The number of partitions of an integer into parts differing by at least 2 equals the number of partitions of that integer into parts which when divided by 5 leave remainder 1 or 4*. The statement of the second identity is similar. For example, there are 6 partitions of 10 into parts differing by at least 2, namely, 10, 9+1, 8+2, 7+3, 6+4, and 6+3+1. There are also 6 partitions of 10 into parts of the form 5m+1 or 5m+4, namely, 9+1, 6+4, 6+1+1+1+1, 4+4+1+1, 4+1+...+1, 1+...+1. The analytic form of the identities is the equality of two infinite series and two corresponding infinite products, the infinite series being the generating function of the partitions satisfying difference conditions, while the products are the generating function for partitions satisfying congruence conditions.

Ramanujan communicated the analytic form of the identities in a letter to Hardy in 1913. Neither Hardy nor any of his contemporaries could prove them. When Ramanujan arrived in England in 1914, Hardy asked him for a proof, but Ramanujan could not supply one. The partition version of the identities is not due to Ramanujan, but due to MacMahon who was completing a book on Combinatory Analysis at that time. So in 1915 when MacMahon's book was published, these partition theorems were stated as unsolved combinatorial problems.

In 1917, while going through some old issues of the London Mathematical Society, Ramanujan came across three papers (1894-96) of the British mathematician L. J. Rogers, who had proved not only these identities but many similar ones. For various resaons, the work of Rogers was largely ignored by his British contemporaries, and Ramanujan's rediscovery of his work brought him due recognition. From then on the name Rogers-Ramanujan identities came to stay.

It must be pointed out, that like Ramanujan, Rogers also only stated the analytic form of the identities and not the partition version. The German mathematician Issai Schur, working independently, had also proved the Rogers-Ramanujan identities and realised their combinatorial significance. This helped Schur obtain the next level partition theorem, now known as Schur's partition theorem.

Ramanujan's motivation for the identities came from the study of an infinite continued fraction. The ratio of the two Rogers-Ramanujan series produces this continued fraction, which has a lovely product representation modulo 5 owing to the Rogers-Ramanujan identities. Ramanujan noticed several beautiful transformation formulas for this continued fraction and used these to calculate the fraction at various special values. Ramanujan's continued fraction is one of the most fundamental objects in the theory of modular forms.

Nowadays by a Rogers-Ramanujan type identity, we mean an identity in the form of an infinite series equals an infinite product, where the series is the generating function for partitions satisfying difference conditions and the product is the generating function for partitions satisfying congruence conditions. Rogers-Ramanujan type identities have arisen in a variety of settings, such as for instance in the study of Lie algebras to problems in statistical mechanics. In the 1980's, the Australian Mathematical Physicist Rodney Baxter first observed that the Rogers-Ramanujan identities arose as the solution of the hard hexagon model in Statistical Mechanics. Subsequently, Andrews and Baxter worked out the complete set of solutions. For this work, Baxter was awarded the Boltzman Medal of the American Physical Society. More recently, Professor Barry McCoy of the Institute of Physics in Stony Brook, in collaboration with Alexander Berkovich, obtained new extensions of various Rogers-Ramanujan type identities by a study of models in Conformal Field Theory in Physics. For this and other work, McCoy was awarded the Heineman Prize in mathematical physics last year.

In the last few years, in collaboration with George Andrews and Basil Gordon, I have developed a new technique called *the method of weighted words* which can be applied to a large class of Rogers-Ramanujan type identities to obtain generalizations and multi-parameter refinements. Subsequently, I also developed a theory of weighted partition identities out of this approach, which provided new relationships between many Rogers-Ramanujan type functions. Thus the study of Rogers-Ramanujan type identities from several points of view will be a major area of research in the years to come.

Mock theta functions: There are many who consider the mock theta functions as among Ramanujan's deepest contributions. They were discovered by Ramanujan in India after his return from England a few months before his death. In his last letter to Hardy, Ramanujan states that he has made a very significant discovery, namely the mock theta functions, and lists several of orders 3, and 5.

Some consider the theta functions as the greatest mathematical discovery of the nineteenth century. Theta functions are extremely interesting because they satisfy transformation properties. The mock theta functions are like the theta functions in the sense that the circle method can be used just as effectively to calculate their coefficients, but they do not satisfy exact transformation formulas. The examples Ramanujan sent to Hardy were well known theta type series but with mild changes in sign which made them mock theta functions. The Lost Notebook of Ramanujan contains several identities involving mock theta functions. In the course of proving many of these, Professor Andrews has explained the partition theoretic significance of several mock theta identities. Recently, Professors Basil Gordon and Richard McCintosh have found some new mock theta functions of orders 8 and 12, and established identities for them. The partition interpretation of these new mock theta identities remains to be explored.

Everlasting beauty: The theory of partitions is just one of the ways in which one can enter the garden of Ramanujan's theorems. Shakespeare said in one of his sonnets "Since brass, nor stone, nor earth, nor boundless sea, but sad mortality oversways their power, how in this rage shall beauty hold a plea, whose action is no stronger than a flower?" Ramanujan's mathematics has an everlasting beauty, and as we move into the new millenium, there is no doubt in my mind, that his influence will continue be strong in several areas of research.