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Department of Computer Science, San Jose State University, San Jose, CA 95192
rucker@cs.sjsu.edu

Ramanujan: Essays and Surveys. Edited by Bruce C. Berndt and Robert A. Rankin. American Mathematical Society (History of Mathematics Series, vol. 22), Providence, 2001, xvi + 347 pp., ISBN 0-8218-2624-7, $79.

Reviewed by Krishnaswami Alladi

How much can be written about the Indian mathematical genius Srinivasa Ramanujan? Quite a bit, for a variety of reasons. First, Ramanujan's mathematical accomplishments are astonishing, and with time we are getting a better understanding of their significance and depth. Next, the fact that Ramanujan made such spectacular discoveries without much formal training makes one wonder how his brilliant mind worked and what motivated his insights. Finally, Ramanujan's life story is a mixture of success and failure, (mathematical) romance and sadness, the mystery of the East and the sophistication of the West. All these things make him a fascinating character to study, with the result that there is a multitude of articles describing various aspects of his life as well as research papers that build on his pathbreaking discoveries. The book under review assembles a fine collection of essays by several distinguished authors on a variety of topics that range over Ramanujan's remarkable life and contributions, his constant struggle with various illnesses in India and in England (none of which dulled his ferocious productivity), his manuscripts and notebooks, and individuals who played an important role in his life and rendered timely help. This book is a worthy sequel to *Ramanujan: Letters and Commentary* [2] (see my review [1]). It also nicely complements Hardy's classic *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work* [3] and Ramanujan's *Collected Papers* [5].

The book opens with a set of four photographs of Ramanujan and an analysis of each photograph. The best known of these is the passport photo of Ramanujan taken when he sailed to England in 1914 to work with G. H. Hardy in Cambridge University. After Ramanujan's death in 1920, Hardy wanted a photograph of Ramanujan in connection with the publication of his collected papers. As related in the book, the young astrophysicist S. Chandrasekhar (who won the Nobel Prize in 1983) was traveling to India, and Hardy requested him to secure a photograph of Ramanujan while he was there. Chandrasekhar met Ramanujan's widow Janaki and was delighted to find that she had preserved the passport photograph of her late husband. Upon seeing this picture Hardy said, "He looks rather ill, but he looks all over the genius he was." (See page 4 in the book under review.) It was this picture that formed the basis for American sculptor Paul Granlund's busts of Ramanujan, created in 1987 for the Ramanujan Centennial Year.

One of the books that inspired Ramanujan was *A Synopsis of Elementary Results in Pure Mathematics* by G. S. Carr. As Hardy said, this was not a great book in any sense, but rather a set of mathematical formulas arranged in a certain logical sequence without proofs. Carr was a tutor in England and he used this book to train his students. This was one of the first mathematics books Ramanujan set eyes on, and he was immediately inspired to prove the formulas there and discover similar formulas. Indeed, Ramanujan recorded his own discoveries in two notebooks in the style of Carr by writing down identities one after the other with no hints of proofs.

Hardy expressed the opinion that Ramanujan did not have a grasp of complex variable theory and that this was the cause for some of the slips that Ramanujan made in the theory of prime numbers. Yet Ramanujan exercised a complete mastery over elliptic and theta functions, a subject in which significant contributions cannot really be made without a firm grip on complex analysis. (Much of his work in that area was done in India prior to his departure for England.) Of Ramanujan's remarkable ability to evaluate elliptic and other definite integrals, Hardy said that if at any time in his lectures he needed the value of a certain integral, he would simply turn towards Ramanujan in the audience, who would provide the answer instantly! We probably can never know how Ramanujan arrived at his formulas because he thought so differently from others with a more conventional training. But we can at least attempt to understand how Ramanujan might have been led to his remarkable findings from a list of books he read. This is the subject of an article by Berndt and Rankin in the volume under review, which includes an analysis of the books Ramanujan studied in India and their influence on him. Among these books was Hardy's *Orders of Infinity,* and it was a problem discussed there concerning the number of primes below a given magnitude that prompted Ramanujan to write to Hardy. The rest is history.

Ramanujan made spectacular contributions to several areas of mathematics at the interface between analysis and number theory, such as elliptic and modular functions, continued fractions, definite integrals, and hypergeometric series. Often his results revealed unexpected connections between fields considered quite distinct. His results are recorded in two notebooks that he maintained in India and in various papers that he published in mathematical journals while in England (1914–1919) and in India before leaving for England. In the book under review there are two thorough articles by Berndt and Rankin on Ramanujan's two notebooks and his various manuscripts.

After his return from England in 1919, Ramanujan was dying, but even in failing health he did not stop doing mathematics. He wrote a letter to Hardy in 1920, just weeks before his death, saying that he had found a new important class of functions called mock theta functions and giving some examples. Ramanujan's work on mock theta functions, now considered to be among his deepest contributions, was written down on loose sheets of paper. After his death, his widow Janaki had the good sense to collect these loose sheets and send them to Hardy. Hardy handed over these last writings of Ramanujan to G. N. Watson in Birmingham and asked him to analyze them. In his Retiring Presidential Address to the London Mathematical Society, Watson gave a lecture entitled "The Final Problem: An Account of the Mock Theta Functions." Watson said,"The study of the five foolscap pages which accompanied the [Ramanujan] letter is the subject which I have chosen for my address. I doubt whether a more suitable title [for my talk] could be found than the title used by John H. Watson, M.D., for what he imagined to be the final memoir on Sherlock Holmes." Watson's lecture is reproduced in full in this book.

After Watson's death, the Ramanujan manuscript mysteriously disappeared. So it acquired the name "The Lost Notebook of Ramanujan." In 1976, George Andrews, one of the foremost experts on the theory of partitions and q-series and on the work of Ramanujan, stumbled across this manuscript while going through the Watson papers at the Wren Library of Trinity College in Cambridge University. Andrews immediately realized the invaluable treasure he had found. The article he wrote in this MONTHLY, which gives a glimpse of the beautiful formulas in the Lost Notebook, is reproduced in the book under review.

In order to gain a complete understanding of Ramanujan, one must also learn about those individuals who played crucial roles in his life and offered timely help. The editors are to be complimented for including biographies of S. Narayana Iyer, who was perhaps Ramanujan's closest mathematical friend in India and supported Ramanujan's mathematical research, and of Mrs. Ramanujan.

Born in 1900, Janaki was wedded to Ramanujan when she was a child, and she was barely a teenager when Ramanujan left for England in 1914. By the time he returned to India in 1919, he was a very sick man, and she tended him with affection and devotion. Although she was uneducated, she was a very intelligent woman and realized that her husband was an unusual genius doing work of great importance. I remember well that in 1987, during the Ramanujan Centennial in Madras, George Andrews paid a wonderful tribute to Mrs. Ramanujan, who was present for the occasion. In a voice choked with emotion, Andrews thanked Mrs. Ramanujan for preserving the pages of the Lost Notebook.

In 1914 Ramanujan published an important paper entitled "Modular Equations and Approximations to π" [4]. This paper contains a myriad of results, including transformation formulas for elliptic and theta functions called modular relations. In fact, Ramanujan discovered more modular relations than Abel, Jacobi, and other such luminaries combined! Hardy was of the belief that Ramanujan did not discover elliptic functions by himself, that he must have had access in India to books from which he learned some of the basic ideas. But Ramanujan's approach to elliptic and theta functions is so original, and his notation so different from that of his illustrious predecessors, that it is hard not to believe that he discovered these results without prior knowledge of the subject.

In addition to modular identities, the paper [4] contains several series representations for the reciprocal of π and for numbers of the form $\pi/\sqrt{n}$, where n is a positive integer. Since these series converge very rapidly, they can be used to calculate the digits of π and other numbers. William Gosper used one of Ramanujan's series for the reciprocal of π to evaluate seventeen million terms in the continued fraction expansion of π. More recently, the brothers David and Gregory Chudnovsky utilized certain extensions of some of Ramanujan's formulas to compute π to about two billion decimal places. The most remarkable thing about this calculation was that the Chudnovsky brothers did it by assembling a computer (by mail order) in their own apartment in New York specifically for this purpose. It is amazing that the work of Ramanujan, who in rural India wrote many of these formulas on a slate and erased them with his elbow, should remain alive in the modern world of the computer!

Many may wonder what is achieved by calculating millions of digits of π. Is it simply for the challenge? (Of the calculation of π to fifteen decimal places, Newton admitted "I am ashamed to tell you to how many figures I carried out these computations, having no other business at this time." See page 192 of the book under review.) In fact, there is more at stake here. Every attempt to understand π has led to new techniques that have applications elsewhere. Often various iterative algorithms are tested by using them to compute the digits of familiar numbers like π. In the article "Ramanujan and Pi" that is reproduced in this book, Jonathan and Peter Borwein have given a detailed account of the historical developments relating to our understanding of π and the usefulness of many of Ramanujan's formulas in the calculation of π.

There are other beautiful and important articles in this book, such as "A Walk through Ramanujan's Garden" by Freeman Dyson and "Ramanujan and Hypergeometric Series" by Richard Askey. I will conclude by briefly discussing some of the contents of Atle Selberg's article "Reflections around the Ramanujan Centenary," which is derived from a talk he gave on December 22, 1987 (Ramanujan's one hundredth birthday), in Madras, India. I was fortunate to be in the audience of this thought-provoking lecture.

Selberg begins by recalling how as a boy he read an article on the Indian genius Ramanujan, and how various astonishing identities of Ramanujan inspired him to study similar questions. In particular, Selberg was intrigued by the famous Hardy-Ramanujan asymptotic series expansion for $p(n)$, the number of partitions of n. As was always the case when Selberg attempted to understand a problem, he set about deriving the Hardy-Ramanujan formula himself. In doing so, he actually ended up with a superior convergent series expansion for $p(n)$, which turned out to have been discovered earlier by Hans Rademacher.

In one of his letters to Hardy in 1913, Ramanujan gave a formula for the coefficients of a certain series expansion of an infinite product that suggested that there ought to be a similar exact formula for $p(n)$. Hardy felt that an exact formula for $p(n)$ in terms of continuous functions was too good to be true but was convinced that it would be possible to construct an asymptotic series expansion. By an ingenious and intricate calculation involving the singularities of the generating function of $p(n)$, Hardy and Ramanujan obtained an asymptotic formula that, when calculated up to a certain number of terms, yielded a value that differed from $p(n)$ by no more than $n^{-1/4}$. Since $p(n)$ is an integer, it is clear that it is the nearest integer to the value given by the series. The series Hardy and Ramanujan obtained was genuinely an asymptotic series in the sense that when summed to infinity it diverges, but—as discovered by Rademacher and Selberg—it can be converted to a convergent series whose sum is $p(n)$ by replacing exponential functions with hyperbolic functions. In the 1913 letter to Hardy, Ramanujan actually used hyperbolic functions to claim an exact formula for a related problem, so he was indeed correct in surmising that a similar exact formula would exist for $p(n)$. According to Selberg, even though Ramanujan claimed the existence of an exact formula for $p(n)$, out of respect for his mentor Hardy he settled for less.

Selberg also expresses the opinion in this article that the great German mathematician Hecke would have been a better mentor for Ramanujan than Hardy. Selberg's observation is based on the fact that many of Ramanujan's important discoveries lay in the area of modular forms, a field in which Hecke was an expert, whereas Hardy was not. But there are other factors that are crucial in mentoring, such as effective communication in a common language, willingness to spend time with a pupil, and above all mutual respect; and Hardy excelled in all these respects. Whether or not one agrees with Selberg's opinion, his article provides many insights and is a magnificent tribute to Ramanujan.

In his charming article "A Walk through Ramanujan's Garden," Freeman Dyson says that any time he felt angry or depressed, he would pull down Ramanujan's *Collected Papers* from the bookshelf and take a quiet stroll in Ramanujan's garden. He recommends this therapy to all who suffer from headaches and jangled nerves. Similarly, this book, with its delightful collection of essays and surveys relating to Ramanujan's life and mathematics, is accessible to, and an inspiration for, laymen, students, and professional mathematicians alike.

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University of Florida, Gainesville, FL 32611
alladi@math.ufl.edu