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*Ramanujan: Letters and Commentary*, by Bruce C. Berndt and Robert A. Rankin, American Mathematical Society, London Mathematical Society, History of Mathematics Series, Vol. 9 (1995), 347 pp., $79.00 (hc), $44.00 (pb).

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Srinivasa Ramanujan is a truly exceptional figure in mathematical history. Born in a poor and orthodox Hindu family in rural India in 1887, Ramanujan did not even have a college degree. Yet, he made startling mathematical discoveries that challenged the finest minds of his and succeeding generations. Ramanujan communicated some of his remarkable findings in letters to G. H. Hardy of Cambridge University, who was so impressed that he helped to make arrangements for Ramanujan to go to England. During the brief period of five years (1914–19) that he spent in England, Ramanujan wrote several fundamental papers, some in collaboration with Hardy, which revolutionized certain areas of mathematics. His work was considered so original and profound that he was made Fellow of The Royal Society (FRS) and Fellow of Trinity College. Unfortunately, the rigors of life in England during World War I combined with his own peculiar habits contributed to a decline in his health. This forced him to return to India in 1919, where he died a year later. Despite his early death, the papers that he wrote in England and in India and the mass of unpublished material that he left behind in the form of two notebooks and loose sheets of paper, have had a deep and lasting impact and continue to inspire researchers today.

Ramanujan's story is sad, yet awe-inspiring in that someone who was so poor financially, who did not have a college education, and who grew up in such an old fashioned society, defied all odds and reached the pinnacle in mathematics, the most abstract and rigorous of all disciplines. Much has been written about Ramanujan and justifiably so, for he and his work are worthwhile studying from many points of view. For the layman, there is the wonderful book by Kanigel [3] describing Ramanujan's incredible life story. For those with a mathematical background, there is an account of Ramanujan's work by Hardy [2] in the form of twelve lectures, considered a classic in exposition. Then there are the Collected Papers [4], where at the beginning there are two charming biographies of Ramanujan written in contrasting styles, one by Hardy and another by P. V. Seshu Aiyar and R. Ramachandra Rao. For those who wish to delve directly into Ramanujan's identities and drink delight of his discoveries, there are the photostat versions of his two Notebooks [5] and The Lost Notebook [6] as well. And to top this all, there is a series of five volumes by Bruce Berndt [1] that thoroughly discuss the hundreds of "entries" made by Ramanujan in his famous notebooks, not to mention many research papers written in this century dealing with Ramanujan's work, in particular by George Andrews. So what could be added of great significance to this already impressive collection? As Shakespeare remarked, would it be like gilding refined gold, painting the lily, or adding another hue to the rainbow? Not at all, as this book under review demonstrates.

By assembling letters to, from, and about Ramanujan, and by giving an authoritative commentary on these letters, Berndt and Rankin have produced a book that should appeal to everyone with an interest in mathematics. Many of the letters contain mathematics. For these letters, Berndt and Rankin explain the results and their significance with appropriate references. There are also letters from persons who knew Ramanujan and had an impact on his life. For such letters there are explanations of the connections these persons had with Ramanujan or his work. By reading the letters along with the comments, we get a better understanding of Ramanujan's personality, his life, and his remarkable work.

This book opens with a series of letters that describe how Ramanujan got a job at the Madras Port Trust and the efforts made by various well wishers to help him pursue his mathematical investigations. Although Ramanujan held a scholarship in his school, owing to his excessive preoccupation with mathematics, he neglected other subjects and consequently failed his college examinations in all subjects except mathematics. Naturally, without a college degree, he could not secure a job to support himself so that he could continue to do mathematics unhindered by financial difficulties. His family was poor and in no position to support him financially. Therefore Ramanujan approached several persons for help and showed them some of the remarkable identities recorded in his notebooks. These persons included Indians in well-placed positions as well as British administrators and professors in educational institutions in the city of Madras. While everyone of them was convinced of Ramanujan's unusual abilities and creativity, no one was able to judge the value of Ramanujan's work or understand it in the proper mathematical framework. One gentleman, R. Ramanchandra Rao, recognized that Ramanujan was doing very original work and gave him some financial support. It was clear that Ramanujan ought to come into contact with first-rate research mathematicians. With this favorable intention in mind, C. L. T. Griffith, Professor of Civil Engineering in Madras, wrote to M. J. M. Hill of the University College, London, about some of Ramanujan's work. Hill realized, as others had, that Ramanujan was very gifted, but when he saw Ramanujan's outrageous claims that

$$1 + 2 + 3 + \cdots = \frac{-1}{12}, \tag{1}$$

and

$$1^2 + 2^2 + 3^2 + \cdots = 0, \tag{2}$$

he felt that Ramanujan had bungled. And in a reply to Griffith, Hill expressed the opinion that these are the kind of blunders one would make without a formal mathematical training. After all, Abel has warned us that "divergent series are in general deadly, and anyone who dares to base a proof on them is doomed to failure". We know now that Ramanujan had not bungled here. On the contrary, these are significant assertions if viewed as follows. Consider the Riemann zeta function, which is defined by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \tag{3}$$

for $\mathrm{Re}(s) > 1$. It is known, and this is a very significant fact, that $\zeta(s)$ admits an analytic continuation as a meromorphic function throughout the complex plane having $s = 1$ as its only singularity. Using this analytic continuation one can show that

$$\zeta(-1) = \frac{-1}{12} \quad \text{and} \quad \zeta(-2) = 0. \tag{4}$$

Observe that by setting $s = -1$ and $s = -2$ formally in (3) and using the values given in (4), the two claims made by Ramanujan in (1) and (2) follow.

But Ramanujan did not explain his claims in this fashion. In fact, he gave no explanation at all and that is what stunned Hill. We know now that Ramanujan had his own theory of infinite series. In particular, to each series, whether it is convergent or not, he associated a "constant". The constant he associated with the series in (1) was $-1/12$ and the constant he assigned the series in (2) was 0. Although no one at that time had any understanding of Ramanujan's methods, it was clear that he was unusually talented. So he deserved to have a scholarship or other financial support to enable him to continue his investigations. Since Ramanujan did not have a college degree, arranging a university scholarship was next to impossible, and so a job was initially secured for him in March, 1912 as a clerk in the Accounts Department at the Port Trust in Madras with the help of its Manager, S. Narayana Aiyar. The Chairman of the Port Trust, Sir Francis Spring, and Narayana Aiyar took a keen interest in Ramanujan's work and later helped him to secure a scholarship at the Madras University and also to prepare for his trip to England. In fact, Narayana Aiyar even worked with Ramanujan on some mathematical problems.

On February 13, 1913, Ramanujan wrote a letter to G. H. Hardy of Cambridge University. This letter is one of the most important and exciting mathematical letters ever written! After begging to introduce himself as a poor clerk in the Port Trust of Madras, Ramanujan gives a collection of amazing results he had obtained. These included representations in the form of series and integrals for the number of primes up to a given magnitude and related arithmetical functions, identities for beta and gamma functions, infinite series identities having arithmetical significance as well as consequences in the theory of elliptic and theta functions, and some unbelievable continued fraction evaluations in terms of certain algebraic numbers. There were some results in the letter that were well known and some that were wrong. But then, there were also many that were startlingly new and very deep. Hardy could prove a few of these, but there were others that defeated him completely. And Ramanujan had not given any hints as to how he went about proving them!

Hardy showed the letter to his colleague J. E. Littlewood and the two came to the conclusion that Ramanujan was a mathematician of the highest caliber and a genius on par with Euler and Jacobi in manipulative ability! Hardy was very excited about this and soon the news spread through Cambridge that at least another Jacobi in the making had been found. There is a very nice letter from Bertrand Russell to his sweetheart Lady Ottoline Morrell that is reproduced in this book wherein Russell says he "found Hardy and Littlewood in a state of wild excitement, because they believe they have discovered a second Newton." In replying to Ramanujan, Hardy pointed out the results that were wrong, those that were well known, and those that were new and most impressive. But he insisted that Ramanujan should supply proofs of his results. In his second letter to Hardy, which contained many more results, Ramanujan says the reason he did not describe his methods was that because they were so unusual, persons with formal training may not appreciate these unconventional approaches (as had been his experience with Hill). But Ramanujan was confident in the validity of his methods because in his first letter to Hardy he actually said—"the local mathematicians are unable to understand me in my higher flights". Both letters of Ramanujan to Hardy are reproduced in full with comments on their mathematical contents.

Hardy was convinced that Ramanujan should not waste any more time in India, but should come to England where his untutored genius would develop its full potential and be given a proper sense of direction. Thus Hardy urged Ramanujan to come to England. Ramanujan initially was reluctant to go abroad owing to his religious background and pressure from his family, but he eventually agreed to go for the sake of mathematics.

During the five years that Ramanujan spent in England he wrote several papers that revealed the range and power of his mathematics. In collaboration with Hardy he wrote two great papers. In the first one, which appeared in 1917, he studied the most commonly occurring values of the number of prime factors of an integer. What is surprising here is that, although prime numbers have been studied since Greek antiquity and various arithmetical functions have been investigated in the subsequent centuries, it was the first systematic discussion of the number of prime divisors of an integer. This paper led to the creation of Probabilistic Number Theory. In another joint paper with Hardy, published in 1918, he gave an asymptotic formula for $p(n)$, the number of partitions of an integer $n$, using the *circle method*. This powerful analytic technique is now the standard tool in Additive Number Theory and its genesis may be traced to a formula that Ramanujan gave for the coefficients $c(n)$ of a certain series in his first letter to Hardy. What Hardy and Ramanujan produced was a series representation for $p(n)$ with terms involving the exponential function. Summing the series up to a certain number of terms dependent on $n$ yielded a value whose nearest integer was $p(n)$. Later, Hans Rademacher made the important observation that replacing the exponential function by suitable hyperbolic functions converts the Hardy-Ramanujan representation into an infinite series that converges to the value $p(n)$. Subsequently, D. H. Lehmer showed that the Hardy-Ramanujan representation in terms of the exponential function is actually divergent as an infinite series. The correspondence between Hardy and Lehmer concerning the partition function is included in this book. It is worth noting that the formula Ramanujan communicated Hardy concerning $c(n)$ made use of hyperbolic functions, as in Rademacher's formula.

In England, Ramanujan also wrote many fundamental papers by himself. In one paper he established unexpected congruence properties for the partition function. After all, partitions represent an additive process and so it is surprising that certain divisibility (congruence) properties are valid here. In yet another famous paper on Modular Equations and Approximations to $\pi$, he gave several astonishing series representations, which are being used in this modern era of computers to calculate the digits of $\pi$. This paper also provided new insights into the properties of elliptic and modular functions.

Although his mathematical productivity was great, Ramanujan never got adjusted to the British way of life. The English winters were too harsh for him and he did not know how to protect himself adequately from the cold. He was a strict vegetarian and so did not want to eat the food served in the dining hall at Trinity College, preferring to cook his own food. But he was not very good at this. In particular, he did not eat a balanced diet. Things were made worse by the wartime rationing in England. All this had a catastrophic effect on his health. He was in and out of several sanatoria and hospitals because he was suspected to have tuberculosis. Even today we are not sure what exactly was Ramanujan's ailment. There is good reason to believe that he was suffering from hepatic amoebiasis, a parasitic infection of the liver or intestines that is found in tropical countries. There are many letters in this book that deal with Ramanujan's health problems.

Because Ramanujan's work was so original and important, Hardy felt that the Indian genius deserved to be made Fellow of The Royal Society and Fellow of Trinity. With Ramanujan's health declining rapidly, Hardy wanted these honors to be conferred without delay. Hardy also felt that this would boost Ramanujan's spirit and have a positive effect on his health. So Hardy moved heaven and earth and finally succeeded in having these honors conferred on Ramanujan. This book contains a wonderful collection of letters between Ramanujan and his relatives and friends in India, wherein Ramanujan describes his mathematical successes in spite of the difficulties of living in England. In these letters he describes the particular Indian vegetarian dishes he is able to prepare. By reading the commentary of Berndt and Rankin, one gets a crash course on the curries and spices of South India! The correspondence between Ramanujan and Hardy while Ramanujan was in nursing homes is also included. It is amazing to see the quality of mathematics that Ramanujan communicated to Hardy from hospital beds in England! Finally, there are letters between Hardy and his British contemporaries giving us an idea of the efforts that went in to get Ramanujan elected Fellow of The Royal Society and Fellow of Trinity.

But Ramanujan's health did not improve substantially as Hardy and others had hoped. As soon as he was well enough to travel, Ramanujan returned to India in 1919. In India, his health worsened, and he died on April 26, 1920. But his mathematical powers had not diminished even in his final moments. He wrote one last letter to Hardy in which he described his latest discovery, the mock-theta functions, and expressed the opinion that these enter mathematics more naturally and beautifully than the false-theta functions of L. J. Rogers. These are now considered to be among Ramanujan's deepest contributions.

After Ramanujan's death, Hardy received the loose sheets of paper on which Ramanujan had scribbled mathematical formulae in his dying moments. We should be grateful to Janaki Ammal, Ramanujan's widow, for not throwing away his final jottings. Later Hardy handed these sheets to G. N. Watson at Birmingham. Contained in these sheets were several deep formulae for the mock-theta functions. Watson wrote two papers on the mock-theta functions but there was much more in these loose sheets that needed to be analyzed. After Watson's death, all the mathematical papers in his house were collected and deposited at the Wren Library in Cambridge University. Included in this collection were the last writings of Ramanujan. Surprisingly, the mathematical world remained unaware of the existence of these notes of Ramanujan until George Andrews accidentally came across them in 1976 while looking through the Watson collection at the Wren Library. The fascinating story of the discovery of Ramanujan's Lost Notebook by Andrews has been told many times. During the Ramanujan Centenary Celebrations in Madras, India, the Lost Notebook and other unpublished papers of Ramanujan as well as some correspondence between Hardy, Watson, and others after Ramanujan's death was brought out in a printed form [6]. These letters along with commentaries are included in this book.

At the beginning of this century Ramanujan was perceived by Hardy as a genius of the first magnitude who, without a sense of direction, had unfortunately wasted some of his best years rediscovering past work in India. Of course, Hardy did acknowledge that there was a significant number of new results in Ramanujan's work and that there were many that were so singularly original that he ranked among the great mathematicians in history. Today we are in a much better position to comprehend the grandeur and significance of Ramanujan's contributions. Hardy said that Ramanujan should have perhaps been born a century earlier during the great days of formulae. But Richard Askey points out that currently in physics there are incredible formulae in several variables that are being analyzed and a genius like Ramanujan would be of invaluable help. As Askey puts it, "the great age of formulae may be over, but the age of great formulae is not!"

Speaking about Ramanujan, Hardy described him as "a poor, uneducated Hindu pitted against the accumulated wisdom of Europe". At this point I would like to compare Ramanujan with another equally astonishing figure, also from India, not in mathematics but in politics, namely, Mahatma Gandhi. Described by Sir Winston Churchill as "the half-naked fakir", Gandhi, wearing only a home spun loin cloth, stemmed the tide of British imperialism by his simplicity, honesty and adherence to non-violence. In a political world of charisma and diplomacy, who would have expected Gandhi to be a success? But Gandhian philosophy has had a lasting impact just as Ramanujan's equations scribbled on a piece of slate or on loose sheets of paper influence the work of some of the most sophisticated mathematicians today. Those turning to Ramanujan for inspiration would find in this book a fine introduction to the type of mathematics that Ramanujan did and a good sample of his great discoveries with references to over three hundred research papers, books, and articles. Those eager to learn about Ramanujan's life would get as a bonus, a glimpse into the culture and traditions of the Hindu way of life in British Colonial India. And finally, what better way to understand the man behind the mathematician Ramanujan than to read letters written by him and about him? Berndt, with the experience he has gained editing Ramanujan's notebooks, and Rankin, one of the veterans in this field, who knew Hardy, Littlewood, Watson, and other British contemporaries of Ramanujan, have combined perfectly to produce this book.

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