to 7th birth anniversary.

RAMANUJAN AND PI

by

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The number pi, the ratio of the circumference of a circle to its diameter, has fascinated mathematicians through the centuries and continues to intrigue researchers even today. Giants like Archimedes, Newton, Euler and Gauss have through their seminal work contributed substantially to our current understanding of properties of pi and shown how this has a direct bearing on many fundamental questions. And to the impressive list of luminaries who have studied this number, Ramanujan's name must be added. As is usually the case when Ramanujan confronts a topic, he provides a touch of magic to it by means of his incredibly beautiful formulae. In a famous paper published in 1914 in The Quarterly Journal of Mathematics (Oxford) entitled "Modular equations and approximations to π", Ramanujan has several tantalising formulae involving pi and other numbers and such expressions are the basis for recent computations of the digits of pi to over two billion decimal places! This paper contains work done by Ramanujan in India prior to his departure to England. It is amazing that Ramanujan, who in rural India wrote many of these formulae on a piece of slate and erased them with his elbow, should remain alive in the modern world of the computer! In this article I will discuss in lay terms some of the important properties of pi and how we understand these in relation to many fundamental problems. In doing so, I will describe some of Ramanujan's observations on pi and how they influence current research. In preparing this article, I profited greatly from the book entitled "Pi and the AGM" by Jonathan Borwein and Peter Borwein as well as from comments by Bruce Berndt.

Early history: The realisation that the ratio of the circumference of a circle to its diameter is the same for all circles, is an important landmark in human history. This invariant number is denoted by the Greek letter π (pi). One finds in the Egyptian Rhind Papyrus, which dates about 2000 BC, the approximate value $(16/9)^2 = 3.1604...$ for π. With regard to the creation of the earth, implicit in The Bible is the statement that π is nearly equal to 3. In attempting to compute the circumference of the earth, the ancients were motivated to calculate π to a high degree of accuracy. Eratosthenes of Alexandria, who is remembered mainly for the *Sieve*, a procedure to generate prime numbers, actually computed the circumference of the earth. But the one figure from that era who towers above every one is Archimedes of Syracuse (287-212 BC). Indeed, Archimedes is considered to be one of the five greatest scientific thinkers of all time along with Newton, Euler, Gauss and Einstein. Archimedes was a master of approximation and of the limit process and in the course of computing the areas and volumes of various geometrical figures, he even anticipated Newton and Liebniz in the development of integral calculus. By computing the lengths of the inscribed and circumscribed polygon of 96 sides for a circle of unit radius, Archimedes showed that π was less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. We realise today, that these early calculations of Archimedes are indeed the first few steps in the harmonic-geometric mean iteration which can be programmed in the computer to give remarkably good approximations to π.

Squaring the Circle: The members of the Pythagorean school believed that all phenomena could be expressed in terms of integers (whole numbers) and rationals (ratios of integers). For those steeped in this philosophy, it must have come as shock when the square root of 2 was shown to be irrational (not rational). The number π also resisted all attempts to produce an exact rational value, increasing the suspicion that it too might be an irrational number. But the irrationality of π was proved only in 1761.

Struggling to understand π geometrically, the Greeks posed the problem of squaring the circle. More precisely, the problem was to construct using only the ruler and compass, a square which is equal in area to a given circle. Since the area of a circle of unit radius is π, what was required was the construction of the side of the corresponding square which will be the square root of π units in length. This is one of the *three problems of antiquity.* The second problem is to trisect any given angle using only the ruler and compass. (It is easy to bisect any given angle and indeed this construction is taught in the early years of high school.) The third problem is to double the cube, that is construct a cube, again with only ruler and compass, which is twice the volume of a given cube. This is equivalent to asking for the construction of the cube root of 2 using ruler and compass. The geometrical construction of the square root of 2 is easy, because this is the hypoteneuse of a right angled isosceles triangle whose equal sides are of unit length, a fact known to any high school student of Euclidean Geometry. All three problems of antiquity are now known to be impossible because of the pioneering work of Galois in the 19-th century, whose study of the solutions of algebraic equations laid the foundations of Group Theory. From the work of Galois it followed that the only numbers which could be constructed using ruler and compass are special types of algebraic numbers. (Algebraic numbers are those which arise as solutions of polynomial equations with integer coefficients.) Both the square root and the cube root of 2 are algebraic numbers but the cube root of 2 is not of the special type. In the case of the problem of squaring the circle, its impossibility is a consequence of the fact that π is not an algebraic number. This was proved by Lindemann in 1882 and is considered to be one of the crowning achievements of 19-th century.

Ramanujan was very much interested in the problem of squaring the circle. In a note published in the Journal of the Indian Mathematical Society (1913), he offered a geometrical construction to obtain an approximation for the square root of π based on the observation that π is nearly 355/113. Ramanujan discusses this approximation again in his famous paper of 1914. The most commonly used rational approximation to π is 22/7 and we understand this now in terms of the continued fraction for π. The number 3 is the first approximation that emerges from the continued fraction, with 22/7 as the second (and better) approximation. The number 355/113 considered by Ramanujan is the third approximation. Each successive approximation from the continued fraction is better than the preceding one. Even before the 15-th century, the Chinese and Indian mathematicians were aware that 355/113 was an extremely good approximation to π.

Representations for pi: The invention of Calculus paved the way for our present understanding of π and other numbers. Between 1665 and 1666 Newton himself calculated π to about 15 decimal places by means of an infinite series for the arc-sine function. His contemporary and rival in mainland Europe, Liebniz, produced in 1674 a more elegant expression for π using the inverse of the tangent function, a fact that was observed independently by the Scottish mathematician James Gregory in 1671. Indeed the Gregory series formed the basis for the calculation of π to 71 decimal places by the British astronomer Edmund Halley and his student Abraham Sharp. Other infinite expressions for π were provided during this period. One of the most beautiful and well known expressions is an infinite product due to John Wallis involving the even numbers in the numerator and the odd numbers in the denominator. In fact Wallis challenged Lord Brouncker by saying "I bet you can't top this!" Lord Brouncker, who was the first president of The Royal Society, was not a mathematician. Nevertheless, he accepted a challenge and produced a lovely continued fraction expansion for π. Since Lord Brouncker did not give a proof of his derivation it remained a mystery as to how he arrived at his result. Bruce Berndt points out that that Ramanujan has several fascinating continued fractions in his notebooks. One of these continued fraction formulas of Ramanujan for a ratio of gamma functions yields Lord Brouncker's fraction as a special case by setting the variable $x = 1$. Interestingly, Ramanujan had communicated this continued fraction along with many other results in his first letter to Hardy in 1913.

Although these infinite expressions for π due to Wallis and his contemporaries were important in understanding many fundamental problems, no one at that time was able to use these formulae to prove that π is irrational. Leonard Euler (1707-83), the most prolific mathematician in history, was the supreme master in the manipulation of infinite expressions. He produced what is perhaps considered to be the most beautiful and important formula in all of mathematics connecting e, the natural base of the logarithms, i, the imaginary square root of minus 1, and π. Euler's formula is that e to the power $i\pi$ equals minus 1. It is said of Euler that he could calculate with as much ease as a fish takes to water or an eagle takes to the wing! (I think the same could be said of Ramanujan). Using his superior powers of calculation, Euler evaluated several infinite series and products in terms of π.

The first proof of the irrationality of π was supplied by Lambert in 1761. Legendre subsequently improved on this and showed that the square of π is irrational, and expressed the opinion that π may not even be an algebraic number, a belief that was shared by Euler. Transcendental numbers, that is numbers which are not algebraic, were not even known to exist at that time. The first transcendental numbers were constructed by Liouville only in 1840. Then in 1873, Charles Hermite showed that e, the natural base of the logarithms, is transcendental. Finally, in 1882, Lindemann, extending the ideas of Hermite, and using Euler's formula, established that π is a transcendental number and thus settled the 2300 year old problem of squaring the circle.

Elliptic and theta functions: There are many who feel that the greatest mathematical discovery of the 19-th century is that of the elliptic and theta functions due primarily to Abel, Jacobi and Weierstrass, each working independently of the others.

Most of us are familiar with the trigonometric functions, sine, cosine, tangent etc. We know that the values of these trigonometric functions repeat, that is they are periodic functions with period 2π. In the study of calculus, it was observed that the values of certain integrals invoving special quadratic polynomials lead to inverses of the trigonometric functions. But even mild variations of these polynomials lead to integrals which are very difficult to evaluate. Such integrals arise for example in the study of the circumference of an ellipse, whose computation is obviously of interest because the planets move around the sun in elliptical orbits. The major realisation was that the inverses of certain of these integrals lead to functions which have two periods, one a real number and another a complex number. These are the elliptic functions. The connection between elliptic and theta functions is that special combinations of theta functions in the form of ratios yield elliptic functions.

Elliptic and theta functions have become very important because of their wide applicability ranging from Statistical Mechanics to Number Theory. The modern notion of an Elliptic Curve in Algebraic Geometry is a far reaching extension of the basic idea of an elliptic function and one that has proved to be extremely important. It is the Theory of Elliptic Curves blended with Number Theory that led to Andrew Wiles' recent proof of Fermat's Last Theorem. Elliptic curves are used today in the fastest algorithms to factor large numbers and to test whether a given large number is a prime number. And elliptic functions and the relations they satisfy, especially some observed by Ramanujan, are the crucial tools employed in the present day calculations of the digits of π.

The AGM: The arithmetic mean (average) of two positive numbers is one half of their sum while their geometric mean is the square root of their product. It is easy to see that the arithmetic and geometric means lie between the two numbers. It is interesting to note that the geometric mean is always less than the arithmetic mean. Exploiting this simple idea, Gauss produced the remarkable arithmetic-geometric mean iteration. More precisely, Gauss starts with two positive numbers a and b with b less than a. Let c and d be their arithmetic and geometric means respectively. Then c and d lie between a and b with d being smaller than c. Gauss then calculates the arithmetic and geometric means of c and d to get numbers which are even closer and repeats this procedure indefinitely. An infinite sequence of pairs is thus generated whose difference keeps shrinking and so these numbers have a limit. This limit is the arithmetic-geometric mean (AGM) of a and b and is denoted by $M(a,b)$. For any number a larger than 1, Gauss evaluated the AGM of a and 1 to be an elliptic integral involving π and the trigonometric function sine, and thus established a connection with the theory of elliptic functions. He then expressed the opinion that this connection would open up a whole new field of analysis.

In the past decade, the AGM and other means obtained by iterative processes have been studied extensively because of their close connection with the elliptic and theta functions and also because these iteration procedures give rapid methods to calculate π and other related numbers. Jonathan Borwein and Peter Borwein have studied such convergence questions for a large class of numbers and especially for π and it was their work which led D.H.Bailey to compute several million digits of π. More recently, Kanada in Japan has employed the iteration techniques of the Borwein's and computed 1.6 billion decimal digits of π.

Ramanujan: Hardy expressed the opinion that Ramanujan did not have a grasp of complex variable theory and that this was the cause for some of the slips that Ramanujan made in the theory of prime numbers. What is most baffling is that Ramanujan had a complete mastery over elliptic and theta functions, a subject in which significant contributions cannot really be made without a firm grasp of complex variable theory. Ramanujan's theory of elliptic functions was work that he did in India prior to his departure to England. Hardy was of the belief that Ramanujan did not invent elliptic functions by himself, that he must have had access in India to Greenhill's book or other books on these topics from which he must have learnt some of the basic ideas. In any case, Ramanujan's work on elliptic and theta functions was work that he did before he was exposed to sophisticated techniques by Hardy. Of Ramanujan's remarkable ability to evaluate elliptic and other definite integrals, Hardy has said, that during the course of his lectures if at any time he needed the value of a certain integral, he would simply turn towards Ramanujan in the audience who would provide the answer instantly!

Ramanujan published an important paper in the Oxford Quarterly Journal of Mathematics (1914) entitled "Modular equations and approximations to π". This paper contains a myriad of formulae including transformation formulas for elliptic and theta functions called modular relations. In fact Ramanujan has discovered more modular relations than Abel, Jacobi and other luminaries combined! Ramanujan's approach to elliptic and theta functions is so original and his notation so different from that of his illustrious predecessors, that contrary to Hardy's opinion, one is inclined to believe that Ramanujan discovered these results without prior knowledge of the subject. Ramanujan's approach to this theory is now gaining acceptance as can be seen from recent lectures by Bruce Berndt entitled "Ramanujan's theory of theta functions".

In addition to modular identities, this paper contains several series representations for the reciprocal of π and for numbers which are of the form π divided by the square root of an integer. Since these series converge very rapidly, it was realised that they could be used to calculate π and other numbers to a high degree of precision. During the past decade, William Gosper used one of Ramanujan's series for the reciprocal of π to evaluate 17 million terms in the continued fraction expansion of π. Within the last two years, the brothers David and Gregory Chudnovsky utilised certain extensions of some of Ramanujan's formulae to compute π to about two billion decimal places. The most outstanding thing about their calculation was that the Chudnovsky brothers did this by assembling a computer (by mail order) in their own apartment in New York - a computer built specifically for this purpose!

Why calculate the digits of pi: Many may wonder what is achieved by calculating millions of digits of π. Is it simply for the challenge? Of his calculation of π to 15 decimal places, Newton admitted "I am ashamed to tell you to how many figures I carried these computations, having no other business at this time." Sir Edmund Hillary's response when asked why he chose to climb Mount Everest was "because it is there!"

In the case of the calculation of the digits of π, there is more at stake than just the challenge. Every attempt to understand π has produced new techniques which have proved applicable elsewhere. The methods developed for studying π shed light on the properties of other numbers. That is also what we hope might happen with regard to the study of *normal numbers*.

A normal number to the base ten is one in whose decimal expansion, every digit from zero to nine occurs with equal frequency, and more generally, any given block of, say, k digits, occurs with frequency ten to the power minus k. The number .12345678910111213 14..., whose decimal digits are simply obtained from writing down all the positive integers, is an example of a normal number to base ten. We know that *almost all* numbers are normal, but it is extremely difficult to prove that a given number is normal. For instance, we suspect that π and e are normal numbers but these questions are at present unresolved.

Another problem is to obtain what is called an *irrationality measure* for π, that is study the degree of approximation of π by rational numbers. In 1958, K.F.Roth of Imperial College, London, was awarded the Fields Medal (the equivalent of the Nobel Prize in mathematics) for showing that all algebraic irrational numbers have irrationality measure equal to 2. We know that *almost all* numbers have irrationality measure 2 including most of the transcendental numbers, but given a specific transcendental number, it is usually very difficult to confirm that its irrationality measure equals 2. In the case of π, it is conjectured that the irrationality measure is 2, but we are far away from this result. A few years ago, the Chudnovsky brothers showed that the irrationality measure for π was less than 16.53.

They obtained such superior irrationality measures for π and related numbers by employing Ramanujan's formulae in his famous paper of 1914.

Ramanujan's ability: Ramanujan's mastery of infinite processes and his superior powers of manipulation are only too well known. It is always fascinating to find out what motivated Ramanujan to write a particular formula down. For instance, in his 1914 paper he offers the fourth root of the number $97\frac{1}{2} - \frac{1}{11}$ as an approximation to π, and provides a geometrical construction. This approximation may also be found in his second and third notebooks but no indication is given as to what led him to this result. One possible explanation (due to N.D.Mermin) is that the decimal expansion of the fourth power of π is 97.409091034002.... and Ramanujan probably observed that the digits 09 appear in succession. So he might have replaced this by the decimal expansion 97.4090909... with the digits 09 repeating indefinitely and thus was led to $97\frac{1}{2} - \frac{1}{11}$ which he wrote in a different form. But this raises the question as to what led Ramanujan to consider the decimal expansion of the fourth power of π in the first place. Bruce Berndt has explained this as follows: "Ramanujan's facility with continued fractions was unequalled in mathematical history. As suspected by Mermin, Ramanujan might have known that the continued fraction for the fourth power of π starts as $97\frac{1}{2}$ and very soon has the large integer 16539 in the sixth step of the expansion (sixth partial quotient). Hence he might have concluded that the fourth power of π should have a very good rational approximation and this probably led him to the decimal expansion."

In summary, trying to understand π is as much of a challenge and pleasure as attempting to understand the mind of Ramanujan. It is only fitting that, Ramanujan, the most romantic mathematical figure in history, should have been charmed by π whose undying beauty has captivated mathematicians from the days of Archimedes to the present!