NOTES

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Two or Three Identities of Ramanujan

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1. Introduction. In his third Notebook [5, pp. 385–6], Ramanujan writes

"If $a/b = c/d$, then

$$(a+b+c)^4 + (b+c+d)^4 + (a-d)^4 = (c+d+a)^4 + (d+a+b)^4 + (b-c)^4$$

and 4 may be replaced by 2 also."

and

$$\begin{aligned}“64\{&(a+b+c)^6 + (b+c+d)^6 - (c+d+a)^6 \\ &-(d+a+b)^6 + (a-d)^6 - (b-c)^6\} \\ \times\{&(a+b+c)^{10} + (b+c+d)^{10} - (c+d+a)^{10} \\ &-(d+a+b)^{10} + (a-d)^{10} - (b-c)^{10}\} \\ = 45\{&(a+b+c)^8 + (b+c+d)^8 - (c+d+a)^8 \\ &-(d+a+b)^8 + (a-d)^8 - (b-c)^8\}^2.” \end{aligned} \qquad (*)$$

Berndt ([1], p. 3) describes this last identity as "an amazing identity" and on p. 102 as "one of the most fascinating identities we have ever seen". He reproduces a proof due to Berndt and Bhargava [2], and refers to proofs by Bhargava [3] and by Nanjundiah [4].

It is possible to verify such identities by using a software package. Thus, for example, we can factor the difference between the left and right sides of $(*)$, and obtain

$$(ad - bc)F$$

where F is a polynomial in a, b, c, d, homogeneous of degree 14, with many terms. This verifies $(*)$, but sheds no light on *why* it is true.

The identities given by Ramanujan are disguised by the fact that they involve *even* powers. The key to understanding the identities is realising that they do not really concern the quantities

$$a+b+c, \quad b+c+d, \quad a-d, \quad c+d+a, \quad d+a+b, \quad b-c$$

but rather the quantities

$$\alpha = a+b+c, \quad \beta = -b-c-d, \quad \gamma = d-a,$$
$$\alpha' = c+d+a, \quad \beta' = -d-a-b, \quad \gamma' = b-c$$

which satisfy

$$\alpha + \beta + \gamma = 0, \qquad \alpha' + \beta' + \gamma' = 0.$$

Thus, Ramanujan's identities are, subject to the condition $ad = bc$,

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha'^2 + \beta'^2 + \gamma'^2,$$
$$\alpha^4 + \beta^4 + \gamma^4 = \alpha'^4 + \beta'^4 + \gamma'^4$$

and

$$64(\alpha^6 + \beta^6 + \gamma^6 - \alpha'^6 - \beta'^6 - \gamma'^6)(\alpha^{10} + \beta^{10} + \gamma^{10} - \alpha'^{10} - \beta'^{10} - \gamma'^{10}) = 45(\alpha^8 + \beta^8 + \gamma^8 - \alpha'^8 - \beta'^8 - \gamma'^8)^2.$$

If, further, we write

$$F_n = \alpha^n + \beta^n + \gamma^n - \alpha'^n - \beta'^n - \gamma'^n$$

the identities become

$$F_2 = 0, \quad F_4 = 0, \quad \text{and} \quad 64F_6F_{10} = 45F_8^2.$$

It is also true that

$$25F_3F_7 = 21F_5^2$$

which, since it involves *odd* powers, does not conform to the pattern of Ramanujan's identities, but is

$$\begin{aligned}25\{&(a+b+c)^3 - (b+c+d)^3 + (c+d+a)^3 \\ &-(d+a+b)^3 - (a-d)^3 + (b-c)^3\} \\ \times\{&(a+b+c)^7 - (b+c+d)^7 + (c+d+a)^7 \\ &-(d+a+b)^7 - (a-d)^7 + (b-c)^7\} \\ = 21\{&(a+b+c)^5 - (b+c+d)^5 + (c+d+a)^5 \\ &-(d+a+b)^5 - (a-d)^5 + (b-c)^5\}^2. \end{aligned}$$

We will prove all these identities. In doing so we can forgo the assumption $ad = bc$ and obtain generalisations of them all. Indeed we will show that with

$$s = 2a^2 + 2b^2 + 2c^2 + 2d^2 + 2ab + 2ac + 2bd + 2cd + ad + bc,$$

$$p = (a+b+c)(b+c+d)(a-d) \quad \text{and} \quad P = (c+d+a)(d+b+a)(b-c)$$

the following hold:

$$F_2 = 6(bc - ad), \quad F_4 = 6(bc - ad)s, \quad 25F_3F_7 - 21F_5^2 = -4725(bc - ad)^2 pP.$$

We establish $(*)$ only for the case $ad = bc$, although more generally

$$\begin{aligned}&64F_6F_{10} - 45F_8^2 \\ &= 9(bc - ad)\{40(p^2 - P^2)s^4 + 15(s^6 + 24(p^2 + P^2)s^3 - 192p^2P^2)(bc - ad) \\ &\quad + 1080(p^2 - P^2)s^2(bc - ad)^2 + 90s(5s^3 - 12(p^2 + P^2))(bc - ad)^3 \\ &\quad + 3888(p^2 - P^2)(bc - ad)^4 + 567s^2(bc - ad)^5 + 2916(bc - ad)^7\}.\end{aligned}$$

2. The Proofs. Let

$$\alpha = a+b+c, \quad \beta = -b-c-d, \quad \gamma = d-a,$$
$$\alpha' = c+d+a, \quad \beta' = -d-a-b, \quad \gamma' = b-c.$$

Then

$$\alpha + \beta + \gamma = 0, \qquad \alpha' + \beta' + \gamma' = 0.$$

We find

$$\alpha^2 + \beta^2 + \gamma^2 = 2q, \qquad \alpha'^2 + \beta'^2 + \gamma'^2 = 2Q$$

where

$$q = a^2 + b^2 + c^2 + d^2 + ab + ac + bd + cd + 2bc - ad,$$
$$Q = a^2 + b^2 + c^2 + d^2 + ab + ac + bd + cd + 2ad - bc$$

Note that

$$q - Q = 3(bc - ad),$$

so if $ad = bc$, $q = Q$.

Since $\alpha + \beta + \gamma = 0$ and $\alpha^2 + \beta^2 + \gamma^2 = 2q$, we have

$$\alpha\beta + \beta\gamma + \gamma\alpha = -q.$$

Similarly,

$$\alpha'\beta' + \beta'\gamma' + \gamma'\alpha' = -Q.$$

Now define

$$p = \alpha\beta\gamma, \qquad P = \alpha'\beta'\gamma'$$

and

$$F_n = \alpha^n + \beta^n + \gamma^n - \alpha'^n - \beta'^n - \gamma'^n.$$

Then

$$\sum_{n\ge0} F_n t^n = \left(\frac{1}{1-\alpha t} + \frac{1}{1-\beta t} + \frac{1}{1-\gamma t}\right) - \left(\frac{1}{1-\alpha' t} + \frac{1}{1-\beta' t} + \frac{1}{1-\gamma' t}\right)$$
$$= \frac{3 - 2(\alpha+\beta+\gamma)t + (\alpha\beta+\beta\gamma+\gamma\alpha)t^2}{1 - (\alpha+\beta+\gamma)t + (\alpha\beta+\beta\gamma+\gamma\alpha)t^2 - \alpha\beta\gamma t^3}$$
$$- \frac{3 - 2(\alpha'+\beta'+\gamma')t + (\alpha'\beta'+\beta'\gamma'+\gamma'\alpha')t^2}{1 - (\alpha'+\beta'+\gamma')t + (\alpha'\beta'+\beta'\gamma'+\gamma'\alpha')t^2 - \alpha'\beta'\gamma' t^3}$$
$$= \frac{3 - qt^2}{1 - qt^2 - pt^3} - \frac{3 - Qt^2}{1 - Qt^2 - Pt^3}$$
$$= 2(q-Q)t^2 + 3(p-P)t^3 + 2(q^2-Q^2)t^4 + 5(pq-PQ)t^5$$
$$+\{3(p^2-P^2) + 2(q^3-Q^3)\}t^6 + 7(pq^2-PQ^2)t^7$$
$$+\{8(p^2q-P^2Q) + 2(q^4-Q^4)\}t^8 + \{3(p^3-P^3) + 9(pq^3-PQ^3)\}t^9$$
$$+\{15(p^2q^2-P^2Q^2) + 2(q^5-Q^5)\}t^{10} + \cdots.$$

Thus we have

$$F_2 = 2(q-Q), \quad F_3 = 3(p-P), \quad F_4 = 2(q^2-Q^2),$$
$$F_5 = 5(pq-PQ), \quad F_6 = 3(p^2-P^2) + 2(q^3-Q^3),$$
$$F_7 = 7(pq^2-PQ^2), \quad F_8 = 8(p^2q-P^2Q) + 2(q^4-Q^4),$$
$$F_9 = 3(p^3-P^3) + 9(pq^3-PQ^3),$$
$$F_{10} = 15(p^2q^2-P^2Q^2) + 2(q^5-Q^5)$$

and so on.

In particular,

$$F_2 = 2(q-Q) = 6(bc-ad),$$

and

$$F_4 = 2(q-Q)(q+Q) = 6(bc-ad)s.$$

If $ad = bc$, $q = Q$,

$F_3 = 3(p-P)$, $F_5 = 5q(p-P)$, $F_7 = 7q^2(p-P)$, and $25F_3F_7 = 21F_5^2$.

More generally,

$$25F_3F_7 - 21F_5^2 = -525pP(q-Q)^2 = -4725(bc-ad)^2pP.$$

Again, if $ad = bc$,

$F_6 = 3(p^2-P^2)$, $F_8 = 8q(p^2-P^2)$, $F_{10} = 15q^2(p^2-P^2)$ and $64F_6F_{10} = 45F_8^2$, which is (∗). The author has further established the last equation of Section 1, but the somewhat lengthy calculations seemed best omitted in a short note.

REFERENCES

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The Bull and the Silo: An Application of Curvature

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1. The Question. This note is inspired by the following problem, which was posed to the Internet newsgroup sci.math a few years ago.

Tethered-bull problem. Suppose a bull is tethered to a silo, whose horizontal cross-section is a circle of radius R, by a leash of length L. What is the area the bull can graze?

The answer, if $L \le R\pi$, is

$$\frac{\pi L^2}{2} + \frac{L^3}{3R}.$$

This can be proved directly by parametrizing the leash using polar coordinates based at the center of the silo, although the details are a bit awkward. In fact, the argument becomes much cleaner if we generalize the problem as follows.

Generalized tethered-bull problem. Suppose a bull is tethered to a silo, whose horizontal cross-section is smooth convex curve of circumference C, by a leash of length $L \le C/2$. What is the area the bull can graze?