FERMAT AND RAMANUJAN - A COMPARISON

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Although Pierre Fermat (1601-65), one of the founding fathers of Number Theory, and Srinivasa Ramanujan (1887-1920), the legendary Indian genius, are separated by centuries, there are many similarities between the two in style and substance. An important part of the legacy of both Fermat and Ramanujan are their many observations recorded informally which have inspired several succeeding generations of mathematicians. Fermat's Last Theorem, which until recently was the most famous unsolved problem in mathematics, was just a marginal entry made by Fermat in a book written by Bachet on the work of the Greek mathematician Diophantus. The statement of the Last Theorem is that for any integer n greater than 2, there are no positive n-th powers which are the sum of two other positive n-th powers. Fermat claimed that he had a truly marvelous proof of this assertion but unfortunately the margin was too small to contain it! This naturally added a spirit of intrigue to the problem. The simplicity of Fermat's Last Theorem belies its depth and difficulty. For the last three centuries, mathematicians both amateur and professional, have attempted to find a proof of Fermat's assertion but did not succeed. These attempts however did yield plenty of new techniques which have proved immensely useful elsewhere. For example, the subject Algebraic Number Theory was born owing to efforts by Kummer and others to understand the unique factorisation property in very general settings, being motivated to study this question while attempting to solve Fermat's Last Theorem. The proof of Fermat's Last Theorem announced by Andrew Wiles in June 1993 and now completed is the culmination of years of effort by many illustrious mathematicians and is the result of the fusion of the Theory of Elliptic Curves and Number Theory. In a similar vein, Ramanujan's incomplete entries in his two notebooks and in the Lost Notebook have engaged mathematicians since the beginning of this century. Several branches of mathematics such as Number Theory, The Theory of Partitions, The Theory of Modular Forms, The Theory of Elliptic and Theta Functions, Hyper-Geometric Series and others have been enriched out of attempts to understand Ramanujan's jottings.

Both Fermat and Ramanujan communicated their wonderful findings in letters. Ramanujan wrote letters to mathematicians in England, desperately seeking recognition for his work. Fortunately, Hardy responded favourably. Fermat communicated regularly with his French peers Pascal and Mersenne among others, as well as with British mathematicians. Fermat's challenge to the British mathematicians was what ultimately led to the complete solution of what is now known as Pell's equation. The name Pell's equation is due to Euler who was under the mistaken impression that the British mathematician Pell had done most of the work on this; but we know now that several centuries earlier, the Indian mathematicians Bhaskara and Brahmagupta had made significant progress on such questions.

There are many number theoretic problems which interested both Fermat and Ramanujan. Fermat stated that every positive integer is a sum of no more than three triangular numbers, four squares, five pentagonal numbers, and so on. Special types of numbers like

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these had been of interest since the days of the Pythagoreans, but no one before Fermat had made such a fundamental observation about them. This assertion of Fermat attracted the attention of many outstanding mathematicians. Gauss gave a proof of the statement that every positive integer is a sum of no more than three triangular numbers, while Lagrange proved the assertion about sums of four squares. The general assertion that every positive integer is a sum of n or fewer n-gonal numbers was established by Cauchy.

Ramanujan was also interested in the representation of integers as sums of squares. But he viewed it from an entirely different angle, in terms of infinite series identities for theta functions. Although Ramanujan seldom stated number theoretic forms of his identities, such interpretations were natural consequences of his results.

Just as Carr's Synopsis, the first book to make an impression on Ramanujan, so strongly influenced his style of writing, Bachet's Diophantus was the book that dominated Fermat's mathematical life. And it was in this book that Fermat copiously made marginal notes, commenting on possible extensions and improvements of results contained therein. In particular, Pythagorean triangles fascinated Fermat.

It is a fact well known to all high school students, that if x, y and z denote the lengths of the three sides of a right-angled triangle with z being the length of the hypoteneuse, then $x^2 + y^2 = z^2$. Pythagorean triangles are those where the x, y and z are integers without a common factor. An example is the triangle with sides 3, 4, and 5. Another example is provided by the triple 5, 12 and 13. There are infinitely many Pythagorean triangles and a formula to generate all of them has been known since antiquity, and can be found in Bachet's book. Fermat was interested in finding whether there were any Pythagorean triangles whose area was also a square and this problem was not discussed by Diophantus or Bachet. Fermat proved by the *method of infinite descent* that such triangles could not exist. The key idea in the method of infinite descent is to show that any positive solution will generate a smaller such solution. By iteration, an infinite sequence of decreasing positive integer solutions would be generated, and this is clearly impossible.

Fermat was not the first person to investigate areas of Pythagorean triangles. As early as the tenth century A.D., the Arabs were interested in determining those numbers which arise as areas of Pythagorean triangles. Fermat proved that squares cannot be areas of Pythagorean triangles. This apparently idle question as to which rational numbers can be realised as areas of right-angled triangles with rational sides has been shown to have significant implications in the modern theory of elliptic curves.

With the exception of The Last Theorem, Fermat is most famous for his method of infinite descent. Fermat used the method of infinite descent to show that various equations do not have solutions among the positive integers, and perhaps believed that this method could be used to give a "truly marvelous proof" of the Last Theorem. In the course of proving that it is impossible to have Pythagorean triangles whose area is a square, Fermat showed that $x^4 + y^4 = z^4$ has no positive integer solutions. He then observed that he could also apply the method of descent on the equation $x^3 + y^3 = z^3$, which is the first case of The Last Theorem. But it was left to Euler to supply the arguments for this case.

In contrast, Ramanujan was interested mainly in equations which had solutions, and especially providing algorithms or formulas for the solutions. The famous Ramanujan taxicab number 1729 is a solution to what appears like a mild variation of the Fermat equation. Indeed 1729 is interesting because $1729 = 12^3 + 1^3 = 10^3 + 9^3$, and this is a solution to

$x^3 + y^3 = z^3 + w^3$. In other words, addition of an extra variable w to the Fermat equation provides a solution. Euler had provided a formula for the general solution to this equation, but in Ramanujan's third notebook there is also a formula for the general solution to the taxi-cab equation which is in some ways more elegant than Euler's.

During the Ramanujan Centennial, Atle Selberg of the Institute for Advanced Study at Princeton pointed out that raising fundamental questions is just as important as solving long standing problems. Both Fermat and Ramanujan have raised several important questions which have engaged mathemations of the highest calibre in the decades that followed. Fermat's assertions interested Euler, who systematically supplied proofs for many of them. And in doing so, Euler noticed improvements and generalisations. Gauss then took over where Euler had left off. Number Theory as it is taught today, is based on Gauss' *Disquisitiones Arithmeticae*, which is the finished product of the foundation laid by Fermat and the structure erected by Euler. Ramanujan's observations have been food for thought since the beginning of this century. At first, Hardy, Watson and Mordell supplied proofs and explanations for many of Ramanujan's observations. In the past few decades, the work of Erdös, Selberg, Deligne, Askey, Andrews, Berndt and others have revealed the grandeur of Ramanujan's discoveries. However, a complete understanding of Ramanujan's writings will take many decades, possibly more than a century.

The importance of the work that has been generated by a study of the writings of Fermat and Ramanujan can be judged by the kind of recognition that the international mathematical community has given to such efforts. In 1986, Gerd Faltings of Germany was awarded the Fields Medal (the equivalent of the Nobel Prize in mathematics for prestige) for proving the Mordell-Weil conjecture. From this work of Faltings it followed that every Fermat equation had at most a finite number of solutions. Faltings' methods have now led to the creation of a new field known as Arithmetic Geometry. Similarly, Andrew Wiles' proof of Fermat's Last Theorem contains many new ideas which will be developed to yield more results in the future. The most famous of Ramanujan's problems was what Hardy called *The Ramanujan Hypothesis* concerning the size of Ramanujan's tau function. Pierre Deligne was honoured with the Fields Medal in 1978 for proving the Ramanujan hypothesis.

Freeman Dyson of The Institute for Advanced Study in Princeton has said that we should be thankful to Ramanujan for not only discovering so much, but also leaving plenty for others to discover! Similarly, we should be grateful to Fermat for raising questions which have kept mathematicians busy for the past three centuries.